Published by Institute of Physics Publishing for SISSA/ISAS

Received: *March 25, 2004*
Revised: *April 14, 2004*
Accepted: *April 16, 2004*

Quintessential Maldacena-Maoz cosmologies

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Abstract: Maldacena and Maoz have proposed a new approach to holographic cosmology based on euclidean manifolds with disconnected boundaries. This approach appears, however, to be in conflict with the known geometric results (the Witten-Yau theorem and its extensions) on spaces with boundaries of non-negative scalar curvature. We show precisely how the Maldacena-Maoz approach evades these theorems. We also exhibit Maldacena-Maoz cosmologies with (cosmologically) more natural matter content, namely quintessence instead of Yang-Mills fields, thereby demonstrating that these cosmologies do not depend on a special choice of matter to split the euclidean boundary. We conclude that if our Universe is fundamentally *anti-de Sitter-like* (with the current acceleration being only temporary), then this may force us to confront the holography of spaces with a connected bulk but a disconnected boundary.

Keywords: AdS-CFT and dS-CFT Correspondence, Differential and Algebraic Geometry, Cosmology of Theories beyond the SM.

Contents

1. The holography of a crunch

A theoretical understanding of the expansion history of the Universe should illuminate two fundamental aspects. The first, of course, is the acceleration (see for example [1, 2]), interpreted theoretically [3] in terms of "de Sitter-like" physics. The second is the possibility that the de Sitter state is metastable, and will ultimately be succeeded by an "anti-de Sitter-like" state terminating in a Big Crunch. The thorny problems associated with the holographic [4] picture of de Sitter spacetime would then be replaced by a new set of challenges: what is the holographic description of an anti-de Sitter Crunch?

There are in fact observational hints [5, 6] raising questions as to whether the Universe has simply evolved from a matter-dominated condition directly to the current vacuum-dominated state; there is some evidence that the evolution has been considerably more interesting than that. A future transition from acceleration to collapse is therefore not as implausible as it may seem from an observational point of view.

From a theoretical point of view, it has long been known [7] that there are arguments which lead to the conclusion that if a de Sitter phase can be realised in string theory at all, this phase can only be metastable. This has been emphasised again in recent work on the cosmological constant problem, for example in [8, 9]. The point has been explained very simply in [10], where it is emphasised that, in a theory with extra dimensions controlled by a radial dilaton, the potential must *vanish at infinity* except for very exotic matter fields. The vanishing of the potential at infinity implies that a de Sitter equilibrium cannot correspond to a global minimum, and this leads either to an eventual catastrophic decompactification (if the potential remains positive) or, perhaps more plausibly, to a transition to contraction

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culminating in a Big Crunch. (The only exception to the statement that the potential vanishes at infinity would be given by "phantom" matter [11] with an equation-of-state parameter below -2. In view of the recent data analyses supporting phantom cosmologies — see for example [5, 12] — this should be investigated: note that while the observational data exclude such low values for the *total* equation-of-state parameter, they do not rule out a mixture of such exotic matter with more normal varieties. But it is certainly not known how to obtain such matter in the string context, so we shall not consider this possibility further.)

The general thesis underlying this "Crunchy" view of cosmic evolution is that our Universe is *fundamentally anti-de Sitter-like rather than de Sitter-like*: the current acceleration is just a passing phase which does not dictate our ultimate fate [13]. That is, the structure of infinity is to be understood in terms of asymptotically anti-de Sitter rather than asymptotically de Sitter spacetimes. Since anti-de Sitter-like cosmologies generically have Bangs and Crunches, the consequences for "holographic cosmology" are obviously profound.

"Anti-de Sitter-like" cosmologies are, by definition, obtained by introducing matter into AdS_4 and allowing it to distort the geometry. The study of such cosmologies from a holographic point of view was recently initiated by Maldacena and Maoz [14], who point out that anti-de Sitter-like cosmologies correspond to euclidean manifolds with a conformal compactification such that the boundary consists of two disconnected components. This immediately opens the way to the use of suitably generalised AdS/CFT techniques, and one might well hope to investigate holographic anti-de Sitter cosmology, and possibly the transition from acceleration to collapse, in this way. (The use of spaces with multiple boundary components to generate *de Sitter*-like cosmologies was explored in [15]. Thus these spaces may be relevant to *both* of the rival candidates for holographic theories of cosmology.)

However, it is well known that Witten and Yau [16] have shown that such euclidean spacetimes give rise to badly behaved field theories on the boundary if the bulk is a geodesically complete Einstein manifold of negative scalar curvature. Maldacena and Maoz avoid this problem by allowing the bulk matter to act on the bulk geometry, so that the bulk metric is no longer Einstein. In essence, the key to understanding cosmological evolution from the AdS/CFT point of view is to take into account this back-reaction, moving beyond treating the bulk as a fixed background.

The prospect of using holography in cosmology is enticing, but it raises many questions. Witten and Yau actually claim that their result still holds even for some bulk manifolds which are *not* Einstein manifolds. It follows that the Maldacena-Maoz cosmologies require bulk matter of some specific kind — it is not enough merely to introduce arbitrary forms of matter, which might still be governed by this more general version of the Witten-Yau theorem. We must therefore ask: what specific properties of the configurations considered by Maldacena and Maoz allow them to avoid the instabilities discussed by Witten and Yau? Furthermore, in their effort to obtain well-behaved field theories on the boundary, Maldacena and Maoz are led to use bulk matter of a kind (Yang-Mills fields) which is not normally considered to be suitable for cosmology.

In particular, this matter satisfies the Strong Energy Condition at all times and cannot describe either the current acceleration or of course the subsequent transition to collapse.

These points might lead one to suspect that the splitting of the euclidean boundary could still be avoided if cosmologically more familiar matter were used instead of the special Yang-Mills configurations considered in [14]. Our objective here is to show that this is *not* the case. We introduce a one-parameter family of cosmological models obtained by inserting *quintessence* (see for example [17]) into AdS_4, instead of Yang-Mills fields. These are Bang/Crunch cosmologies which nevertheless have temporarily accelerating phases; they therefore yield a very simple model of the transition from acceleration to collapse. Furthermore, the euclidean version has a disconnected boundary, precisely as in [14]. Using these, we can explain precisely what properties bulk matter should have in order to evade the Witten-Yau theorem. (We are not claiming to have solved the difficult problem of obtaining quintessence from string or M-theory; we must assume that this is possible, perhaps along the lines indicated in [18] or [19, 20]. If that can indeed be done, then one expects the quintessence to have a well-behaved description in terms of a field theory configuration on the euclidean boundary.)

Our conclusion is rather surprising: it is actually quite easy to avoid the strictures of the Witten-Yau theorem, even in its strongest version (due to Cai and Galloway [21]). We conclude that cosmological models with euclidean versions having multiple boundaries (henceforth, "Maldacena-Maoz cosmologies") do not in general lead to unacceptable physics. Furthermore, they are "generic" in the sense that they do *not* require the use of the interesting but (in the cosmological context) somewhat unusual bulk matter studied in [14]. They therefore force us to confront the apparent conflict with holography which arises when one apparently has *two* independent field theories associated with *one* bulk. This truly fundamental puzzle cannot, in short, be disposed of by claiming that it cannot arise in physically realistic circumstances.

(We note before proceeding that the reader should not confuse the wormholes considered by Maldacena and Maoz with lorentzian wormholes [22]. Only the euclidean versions of the Maldacena-Maoz spaces have wormholes. They also differ from the AdS wormholes studied in [23], which have the local geometry of (euclidean) AdS itself, and which have to be sustained by a brane at the wormhole throat; though [23] was also motivated by a wish to investigate the "disconnected boundary" problem.)

2. Anti-de Sitter spacetime and its crunchy relatives

The Maldacena-Maoz cosmologies, and their rather subtle relationship with anti-de Sitter spacetime itself, can be understood with the help of the points raised in the following discussion.

Anti-de Sitter spacetime is of course not very interesting as a cosmological model. Written in FRW form, its metric is (in four dimensions) given by

$$g^{-}(AdS_4) = -dt^2 + \cos^2\left(\frac{t}{\mathrm{L}}\right)\left[dr^2 + \mathrm{L}^2 \sinh^2\left(\frac{r}{\mathrm{L}}\right)[d\theta^2 + \sin^2(\theta)d\phi^2]\right], \qquad (2.1)$$

where the cosmological constant is $-3/\mathrm{L}^2$, and where a negative superscript will always indicate a lorentzian metric, a positive superscript denoting a euclidean metric. Notice that the spatial sections are just copies of the three-dimensional hyperbolic space H^3 of sectional curvature $-1/\mathrm{L}^2$, with the metric expressed in terms of polar coordinates. Thus (2.1) *appears* to represent a Bang/Crunch cosmology with negatively curved spatial sections: a strange mixture of the traditional "closed" and "open" cosmological models. It also *appears* to represent a time-dependent geometry, as is normal in cosmology. In reality, the apparent spacelike singularities at $\mathrm{t} = \pm\pi/2$ are mere coordinate singularities, which arise because all of the timelike geodesics perpendicular to the spatial surface at $\mathrm{t} = 0$ intersect periodically. The apparent time-dependence is likewise illusory, since the full anti-de Sitter geometry has a timelike Killing vector, which may be thought of as arising from the Killing spinors associated with the AdS supersymmetries. (This is the counterpart of the fact that the de Sitter spacetime, which has no timelike Killing vector, can be made to appear static by means of a choice of coordinates.) See [24] for a good discussion of these peculiarities of anti-de Sitter spacetime.

The reason that AdS_4 can, despite appearances, avoid being a Bang/Crunch spacetime, is essentially that it contains nothing apart from the matter which supplies the (negative) cosmological constant — this would be p-form matter in the string theory context. This simplifies the structure of the curvature tensor to the extent that AdS_4 fails to satisfy the *generic condition* in the Hawking-Penrose cosmological singularity theorem ([25, p. 266]). We can therefore expect that the introduction of matter into AdS_4 will cause the geometry to satisfy the generic condition and so produce a singular cosmological spacetime of the kind we are seeking, since AdS_4 does satisfy all of the other conditions of the singularity theorem, *including* the Strong Energy Condition. (Strictly speaking, it does not have a compact edgeless achronal set as the relevant version of the singularity theorem requires, but by taking the quotient by a freely acting group which compactifies the spatial sections, we obtain a spacetime which does have such sets; this spacetime is still non-singular, so we see that it is indeed the generic condition which is the relevant one here. Notice that anti-de Sitter spacetime differs in this regard from de Sitter spacetime, which violates *two* conditions of the Hawking-Penrose theorem, namely the generic condition *and* the Strong Energy Condition. Thus the introduction of small amounts of matter into *de Sitter* spacetime should not be expected to cause the spacetime to become singular.) We now consider a simple example illustrating this point. (For a very different approach to obtaining cosmological spacetimes from anti-de Sitter spacetime, see [26].)

The current observational evidence does not rule out negatively curved spatial sections, but the sections are close to being flat. Let us take the AdS_4 metric and simply replace the negatively curved spatial sections with flat ones. For reasons which will become apparent, we take the flat sections to be compact, (say) cubic tori. For Bang/Crunch cosmologies there will be a toral spatial section of maximum size (at $t = 0$); we shall specify that the circumferences of that torus shall be $2\pi\mathrm{A}$, for some suitably large constant A. (Other compact flat manifolds are equally acceptable, though of course one may prefer to impose orientability.)

Modifying the AdS_4 metric in this way, we obtain

$$g^{-}(1,A) = -dt^2 + A^2\cos^2\left(\frac{t}{\mathrm{L}}\right)[d\theta_1^2 + d\theta_2^2 + d\theta_3^2]\,, \tag{2.2}$$

where the torus is parametrised by angles and where the notation $g^{-}(1,A)$ will be explained below. Unlike the anti-de Sitter metric, this metric is genuinely singular: it has a Big Bang at $\mathrm{t} = -\pi/2$ and a Big Crunch at $\mathrm{t} = +\pi/2$, as we shall prove explicitly later. Of course, it is not like AdS_4, which solves the Einstein equation with no matter apart from that which generates the negative cosmological constant: we have introduced matter (of a kind to be described below) into an anti-de Sitter background. The effect of this matter is to flatten the spatial sections, to introduce spacelike singularities at $\mathrm{t} = \pm\pi/2$, and also to remove the timelike Killing vector. (This last follows from the fact, to be established below, that the coordinates in (2.2) cover the entire spacetime.) One can understand this physically by thinking of the *negative* cosmological constant as being associated with an "attractive force" which increases with separation. As soon as we introduce matter into AdS_4, this "attraction" inevitably results in a Crunch. In this sense, $g^{-}(1,A)$ is "more generic" than the pure anti-de Sitter metric. (Like AdS_4, this spacetime satisfies the Strong Energy Condition (see below), but it also satisfies the generic condition and it has compact achronal edgeless sets because the spatial sections are compact; and so it has to be singular by the Hawking-Penrose theorem.)

Now the euclidean version of AdS_4 is of course the hyperbolic space H^4, the four-dimensional simply connected space of constant negative curvature. As is well known from studies of the AdS/CFT correspondence [27], the conformal boundary of H^4 is a conformal three-sphere, S^3, which is compact and connected. When however we consider the euclidean version of $g^{-}(1,A)$, given by

$$g^{+}(1,A) = dt^2 + A^2\cosh^2\left(\frac{t}{\mathrm{L}}\right)[d\theta_1^2 + d\theta_2^2 + d\theta_3^2]\,, \tag{2.3}$$

we see immediately that, at least in the most obvious interpretation, the conformal boundary of the underlying manifold is compact but *not* connected: it consists of two copies of the torus, T^3. The introduction of matter into AdS_4 has not just flattened the conformal boundary: in the euclidean picture *it has split it into two connected components.* (We chose the flat sections to be compact so that the conformal boundary should be compact, thereby avoiding all of the complications which arise in AdS/CFT if the boundary is allowed to be non-compact. This also has the benefit of making it clear that the boundary is indeed disconnected — this is sometimes far from evident when the sections are non-compact, as for example in the foliation of AdS_5 by AdS_4 slices. (See the discussion in the Conclusion, however.) We stress that any compact flat three-dimensional manifold would be as suitable as the cubic torus we are using here.)

Maldacena and Maoz [14] propose a more general version of this construction as a way of understanding Bang/Crunch cosmologies from the AdS/CFT point of view. They propose to study the holography of general euclidean manifolds of the form

$$g^{+}(\mathrm{F},\Sigma) = dt^2 + \mathrm{F}^2(t)\,g(\Sigma)\,, \tag{2.4}$$

where F is a nowhere-zero function which resembles $e^{|t|/L}$ as t tends to $\pm\infty$ (see section 6 for a precise version of this), where L is some positive constant, and where Σ is some compact three-manifold with euclidean metric $g(\Sigma)$. Such manifolds have Bang/Crunch cosmologies as their lorentzian versions, while the euclidean version locally resembles euclidean anti-de Sitter spacetime near $t = \pm\infty$. (That is, the sectional curvatures all asymptotically approach $-1/L^2$; this is true whatever the geometry of Σ may be.) Clearly there is an opportunity to bring AdS/CFT techniques to bear on Bang/Crunch cosmologies in this way. However, it is also clear that the conformal compactification of the euclidean version has a boundary which consists of *two* copies of the compact manifold Σ.

This suggestion, therefore, immediately forces us to confront one of the deepest problems in holography: how does the holographic philosophy deal with a situation in which there are *two* boundaries, inhabited by two (presumably) distinct field theories, but only one bulk? The correlators between the two boundaries should factor from the point of view of the field theory, but not from the point of view of the bulk: a flagrant violation of the holographic principle. This serious problem was pointed out by Witten and Yau [16], who suggested an ingenious solution which we shall now explain.

3. The Witten-Yau theorem

Witten and Yau proposed to deal with the problem of disconnected boundaries in the most radical way, by attempting to prove that such a situation cannot arise in a physically reasonable manner. They showed that

(*a*) if the bulk is a geodesically complete connected euclidean Einstein manifold of negative scalar curvature, and

(*b*) if the conformal structure induced on any component of the boundary is represented by a metric of positive constant scalar curvature,

then the conformal boundary *must be connected.* (Simplified proofs, with many related results, were given in [21] and [28] (see also [29]); another proof, with the slightly stronger hypothesis that the scalar curvature should be positive on *all* components of the boundary, was given in [30].)

The condition that the scalar curvature on the boundary should be positive can be partly justified by noting that, in the negative case, there is a non-perturbative instability arising from the nucleation of branes in the bulk; negative scalar curvature at infinity implies that the action is decreased as the brane moves towards the boundary [31]. (For a recent discussion of this, and of the difficulties which arise when one attempts to suppress the instability, see [32].) As it stands the theorem does not explain what happens in the case where the boundary has zero scalar curvature; indeed, in general the physical acceptability of this case is not fully understood, but certainly we must allow the special case in which the boundary is completely flat (and not just scalar-flat). Fortunately the Witten-Yau theorem was improved by Cai and Galloway [21] so that the same conclusion can be reached but with the condition on the boundary scalar curvature weakened to "non-negative" instead

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of "positive". The upshot is that, in physically reasonable cases, the boundary *cannot* be disconnected if the bulk is a complete euclidean Einstein manifold of negative scalar curvature. Thus the "two boundaries/one bulk" conundrum cannot arise in that case.

Taking the bulk to be an Einstein manifold means that we are ignoring the effect of bulk matter on the bulk geometry. That is a good approximation in some circumstances, but not in all — certainly not in cosmology. So for anti-de Sitter-like cosmologies the question returns: can there be non-perturbatively stable spacetimes with two boundaries and one bulk if the bulk matter is allowed to act on the bulk metric, so that it is no longer an Einstein metric? Witten and Yau suggested an answer in this case also. They argued that their result continues to hold if the Einstein condition is weakened in the following way. Think of the Ricci tensor as a (1,1) tensor, so that its eigenvalues are well-defined; they are functions of position in general. If the eigenvalue functions are $\mathrm{Ric}_{(j)}$, where j ranges from 0 to 3 in four dimensions, then the Einstein condition is just

$$\mathrm{Ric}_{(j)} = -\frac{3}{\mathrm{L}^2}\,, \qquad j = 0, 1, 2, 3\,. \tag{3.1}$$

Witten and Yau weaken this to the condition that the eigenvalues of the Ricci tensor should be bounded below *everywhere in the bulk* by their asymptotic values as the boundary is approached. If the asymptotic sectional curvature is $-1/\mathrm{L}^2$, then the condition replacing (3.1) is just

$$\mathrm{Ric}_{(j)} \geq -\frac{3}{\mathrm{L}^2} \qquad j = 0, 1, 2, 3\,. \tag{3.2}$$

In short, the boundary still has to be connected as long as the back-reaction of bulk matter always tends to *increase* the Ricci curvature. (Allowing the Ricci eigenvalues to become functions of position, however, immediately raises questions as to what exactly happens to these functions as infinity is approached. This subtle point was raised by Cai and Galloway [21], who stressed the importance of the *rate* at which the Ricci eigenvalues approach the asymptotic values. We shall explain this in detail below.)

Witten and Yau state that their condition on the Ricci curvature corresponds to having matter fields excited in an asymptotically anti-de Sitter space. (The asymptotic values of the Ricci eigenvalues are interpreted as the anti-de Sitter cosmological constant.) The first question is then: what kind of matter would correspond to a geometry with such Ricci eigenvalues? Secondly we should ask: is it physically reasonable to impose the conditions demanded by Cai and Galloway on the asymptotic data? For an asymptotically anti-de Sitter black hole, there are well-motivated conditions on the rate at which the metric should approach the AdS_4 metric [33], but in general one would not expect black hole boundary conditions to be relevant to cosmology.

In the examples considered by Maldacena and Maoz, the bulk matter is typically a Yang-Mills instanton or meron, and, precisely in order to evade the "Einstein" version of the Witten-Yau theorem, this matter is allowed to deform the bulk geometry so that the bulk is *not* an Einstein manifold.

In the case of the meron, one begins with the compactification of eleven-dimensional supergravity on S^7. This has a consistent truncation to a theory with an SU(2) gauge field and a graviton. Maldacena and Maoz consider a gauge field with a lagrangian density

$$\alpha\sqrt{g}\, F^a_{\mu\nu} F^{a\mu\nu}\,, \tag{3.3}$$

where α is a non-negative constant. Adding this to the Einstein-Hilbert lagrangian density and including a negative cosmological constant $-3/\mathrm{L}^2$ (since we want an "AdS$_4$-like cosmology", that is, one which reverts to AdS$_4$ if α is zero), Maldacena and Maoz solve the resulting field equations and obtain the euclidean metric

$$g^+_{MM} = dt^2 + \mathrm{L}^2\left[\left(\alpha+\frac{1}{4}\right)^{1/2}\cosh\left(\frac{2t}{\mathrm{L}}\right)-\frac{1}{2}\right]\times[d\chi^2+\sin^2(\chi)\{d\theta^2+\sin^2(\theta)d\phi^2\}]\,, \tag{3.4}$$

defined on a manifold with universal cover of the form $\mathbb{R}\times \mathrm{S}^3$. The lorentzian version,

$$g^-_{MM} = -dt^2 + \mathrm{L}^2\left[\left(\alpha+\frac{1}{4}\right)^{1/2}\cos\left(\frac{2t}{\mathrm{L}}\right)-\frac{1}{2}\right]\times\left[d\chi^2+\sin^2(\chi)\{d\theta^2+\sin^2(\theta)d\phi^2\}\right], \tag{3.5}$$

is indeed a Bang/Crunch cosmology with locally spherical spatial sections: computing the invariant

$$\left[\mathrm{R}^{\mu\nu}+\frac{3}{L^2}g^{\mu\nu}\right]\left[\mathrm{R}_{\mu\nu}+\frac{3}{L^2}g_{\mu\nu}\right] = \frac{12\alpha^2/L^4}{[(\alpha+1/4)^{1/2}\cos(2t/L)-1/2]^4}\,, \tag{3.6}$$

we see that the singularities in g^-_{MM} at times t_B, t_C, given by

$$-\,t_B = t_C = L\cos^{-1}\left(\frac{1}{\sqrt{1+4\alpha}}\right), \tag{3.7}$$

are genuine curvature singularities at a Bang and a Crunch respectively. Notice that the total proper lifetime of the Maldacena-Maoz universe is $2L\ \cos^{-1}(1/\sqrt{1+4\alpha})$, which becomes *shorter* as α is reduced. One can imagine that the Yang-Mills fields are "holding apart" the Bang and the Crunch. The full extent of conformal time for this spacetime,

$$\int_{t_B}^{t_C}\frac{dt}{\sqrt{(\alpha+1/4)^{1/2}\cos(2t/L)-1/2}} \tag{3.8}$$

is always less than πL; for example, it is about 2.384×L if $\alpha = 0.75$. As the covering spacetime is conformal to part of the Einstein static universe [25], the precise shape of the Penrose diagram will depend on the choice of topology for the spatial sections: if they have the topology of S^3, then the diagram will be a rectangle which is wider than it is high, meaning that (unlike, for example, in a matter-dominated FRW cosmology with spherical spatial sections), the particle horizons never disappear, even during the contraction phase. On the other hand, if the spatial sections are (for example) copies of $\mathbb{RP}^3$ (see [34]) then this will not be so; this is of course a possibility consistent with the metric (3.5). The Penrose diagram will have the form shown in figure 1 if we choose $\alpha = 0.75$ and $\mathbb{RP}^3$ spatial sections. (The diamonds on the right indicate that points there represent copies of $\mathbb{RP}^2$ rather than two-spheres.)

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By inspecting this diagram we see that there is no way that the spacetime can be extended. Therefore the coordinates used in (3.5) are global coordinates, and it follows that there is no timelike Killing vector here — the geometry is genuinely time-dependent. (By contrast, a Penrose diagram of the region of AdS_4 covered by the coordinates used in (2.1) would immediately reveal that this region can be extended, and that there is a timelike Killing vector once this extension is performed.)

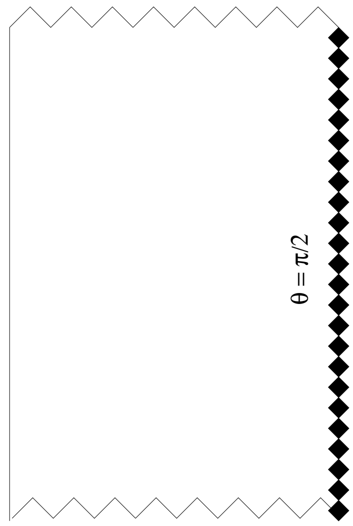

Figure 1: Penrose diagram of MM spacetime, $\alpha = 0.75$, $\mathbb{RP}^3$ sections.

The reader has no doubt observed that the Maldacena-Maoz spacetime is very different indeed to the AdS_4 from which it originates: it is singular, globally hyperbolic, and has a time-dependent geometry, while AdS_4 has none of these properties. In fact, the Einstein equations for FRW spacetimes imply that, because the spatial sections here are positively curved, the energy density of the Yang-Mills fields must be greater (at all times) in absolute value than the energy density contributed by the cosmological constant. Thus, the Maldacena-Maoz spacetime is not a "small perturbation" of AdS_4. Nevertheless, the euclidean version of this spacetime does have almost the same asymptotic geometry as the euclidean version of AdS_4 — indeed, at sufficiently large distances, the only real difference is precisely the fact that the euclidean Maldacena-Maoz spacetime has two asymptotic regions. This is the key virtue of the Maldacena-Maoz proposal: even though Bang/Crunch cosmologies are vastly different from AdS_4, their euclidean versions are sufficiently similar as to warrant hope that a holographic description is possible.

Returning to the euclidean version given by equation (3.4), note that if the meron is turned off by setting $\alpha = 0$, then a simple calculation shows that this is just the standard metric for four-dimensional euclidean anti-de Sitter space. That is, we obtain ordinary hyperbolic space H^4, with the metric expressed in polar coordinates, if the sections are copies of S^3. (If the sections are copies of $\mathbb{RP}^3$, then we obtain an orbifold of H^4, but this orbifolding only happens if α is *exactly* zero.) If α does not vanish, then the locally spherical sections do not shrink down to zero size at $t = 0$ as they do in H^4. Instead they open up again to a second region like the first. (Geometrically this is rather like a smooth version of the wormhole constructed in [23]. The physical difference is that that wormhole required exotic matter in the form of a negative-tension brane at the throat, the bulk being otherwise matter-free.) The conformal boundary consists, again in the most obvious interpretation, of two copies of some three-manifold with the local geometry of S^3, which of course has a metric of positive scalar curvature. The bulk is geodesically complete, but it is not an Einstein manifold, so the simplest version of the Witten-Yau theorem, which assumes that the bulk is an Einstein manifold, does not apply here.

As we saw, however, this alone is not enough: the Witten-Yau theorem can handle *some* non-Einstein manifolds. Indeed, the Witten-Yau stipulation that the bulk matter should increase the Ricci curvature (that is, it should make the Ricci curvature less negative than it is in anti-de Sitter spacetime) *seems* very reasonable — it looks very much like a euclidean version of the Strong Energy Condition (which just requires that a given form of matter should have an energy density making a non-negative contribution to the Ricci curvature). In fact, the relevant Ricci component for g^{-}_{MM} is

$$\mathrm{R}_{00}(g^{-}_{MM}) = \frac{3}{L^2} + \frac{3\alpha/L^2}{[(\alpha+1/4)^{1/2}\cos(2t/L) - 1/2]^2}, \tag{3.9}$$

and we see explicitly that the Yang-Mills field in the lorentzian Maldacena-Maoz cosmology does indeed make a positive contribution to the Ricci curvature. (Actually, in agreement with the general discussion above, a simple calculation shows that, at all times, its contribution is larger than that of the cosmological constant itself.) In fact, Yang-Mills fields always satisfy the Strong Energy Condition. One might have expected the Witten-Yau theorem to forbid a double boundary here; but evidently it does not. What is going wrong?

There are actually *two* things "going wrong" here, and it is important to be clear about this, because one of the problems is more important than the other. Let us explain.

4. Escaping the menace of the WY theorem, part 1

The first reason that the Witten-Yau theorem (even in the version which does not require the bulk to be an Einstein manifold) does not apply to the Maldacena-Maoz manifold is that matter which satisfies the Strong Energy Condition *does not necessarily cause the Ricci curvature to increase in all directions of the euclidean version.* The reason for this can be seen in the following elementary way.

Consider a euclidean field theory in (n+1) dimensions with an energy-momentum tensor $\mathrm{T}_{\mu\nu}$. Diagonalising, we can express T with respect to an orthonormal basis as

$$\mathrm{T}_{\mu\nu} = \mathrm{diag}(p_0, p_1, p_2, \ldots, p_n)\,, \tag{4.1}$$

where the p_i are the eigenvalues. (Henceforth, Greek letters are spacetime indices; all other indices are just labels, as in (3.1) and (3.2) above.) The Einstein equation (with cosmological constant) gives us Ricci eigenvalues

$$\mathrm{Ric}_{(i)} = -\frac{n}{L^2} + p_i - \frac{1}{n-1}\sum_{j=0}^{n} p_j\,. \tag{4.2}$$

From this we immediately see that the condition for the introduction of matter into AdS_{n+1} to increase the Ricci eigenvalues is just

$$p_i \geq \frac{1}{n-1}\sum_{j=0}^{n} p_j \qquad \forall i\,, \tag{4.3}$$

or

$$p_i \geq \frac{1}{n-2}\sum_{j\neq i}^{n} p_j \qquad \forall i\,. \tag{4.4}$$

To see what this means, consider the five-dimensional case ($n = 4$), so that we have

$$p_0 \geq \frac{1}{2}(p_1 + p_2 + p_3 + p_4)\,, \tag{4.5}$$

and likewise

$$p_1 \geq \frac{1}{2}(p_0 + p_2 + p_3 + p_4)\,. \tag{4.6}$$

Combining these we have

$$p_0 \geq p_2 + p_3 + p_4\,, \tag{4.7}$$

and similarly with the roles of p_0 and p_2 reversed, whence it follows that $\mathrm{p}_3 + \mathrm{p}_4$ must be non-positive, and of course the same applies to any distinct pair of eigenvalues. This means that of the p_i, at most one can be positive. Clearly a similar argument works in all dimensions. But this is an unreasonably restrictive requirement; for example, it is easy to construct Yang-Mills configurations, in any euclidean dimension, such that the energy-momentum tensor has more than one positive eigenvalue — see below for an example. The Witten-Yau inequalities (3.2) therefore do not apply to such fields, despite the fact that the lorentzian versions satisfy the Strong Energy Condition. In short, the inequalities (3.2) cannot in general be motivated by imposing this energy condition; in fact, they apparently require that the SEC be violated.

In the specific case of the euclidean metric (3.4) studied by Maldacena and Maoz, the eigenvalue of the Ricci tensor corresponding to the coordinate t may be computed as

$$\mathrm{Ric}_{(0)} = -\frac{3}{L^2} - \frac{3\alpha/L^2}{[(\alpha+1/4)^{1/2}\cosh(2t/L) - 1/2]^2}\,, \tag{4.8}$$

and since the Yang-Mills parameter α is non-negative, we see at once that this particular eigenvalue is in fact *decreased* by the presence of the matter. (The other three are increased; since the Yang-Mills energy-momentum tensor is traceless in four dimensions, the Ricci tensor is proportional to this energy-momentum tensor, and so these three positive contributions mean that there are three positive eigenvalues of the energy-momentum tensor, again showing, in view of our earlier more general discussion, that the inequalities (3.2) are not satisfied here.)

Thus we see explicitly that the Maldacena-Maoz manifold violates the Witten-Yau inequalities. To see this in a more dramatic way, notice that at the wormhole throat ($t = 0$) we have from (4.8) that

$$\mathrm{Ric}_{(0)}(\mathrm{Throat}) + \frac{3}{L^2} = \frac{-3[(\alpha+1/4)^{1/2}+1/2]^2}{\alpha L^2}, \tag{4.9}$$

from which we derive the interesting fact that if α is very small, so that the geometry is almost indistinguishable from that of euclidean AdS_4 except very near to the throat, then the extent of the violation of the Witten-Yau inequalities is very *large*, not small: the inequalities are *not* close to being satisfied (near the throat) in this case.

For Yang-Mills fields in four dimensions, the situation we have been discussing is in fact generic: the "energy-momentum tensor" in that dimension must be traceless, and so the same is true of its contribution to the Ricci curvature. Since the sum is zero, there must, in any non-trivial configuration, be both positive and negative contributions, and so it is clear that the inequalities (3.2) cannot possibly be satisfied here. Thus the Witten-Yau theorem does not apply to *any* non-trivial Yang-Mills configuration in a four-dimensional asymptotically anti-de Sitter spacetime. (In [23], the Witten-Yau theorem was avoided in a different way: the conditions (3.2) are satisfied everywhere except at a negative-tension brane, but, because of the presence of the brane, the bulk is not geodesically complete, and this too renders the Witten-Yau theorem inapplicable.)

To summarize: the mere fact that matter has been inserted into AdS_4 does not explain the ability of the Maldacena-Maoz cosmologies to have disconnected euclidean boundaries with positive scalar curvature: the presence of matter is necessary but not sufficient. The simplest reason for this ability — though, as we shall see, even this is far from the full explanation — is just that, contrary to intuition, the well-behaved matter used in [14] violates the Witten-Yau inequalities.

The question now is this: suppose that we consider other forms of bulk matter. Do these, too, naturally violate the inequalities (3.2), or is this a specific property of Yang-Mills fields? If the latter is the case, then one might try to ascribe the disconnectedness of the boundary to a cosmologically unrealistic choice of bulk matter.

In fact, the true situation is much more complex. To understand why, consider scalar matter instead of Yang-Mills fields. Here, for a scalar field φ with the usual kinetic term and with potential $\mathrm{V}(\varphi)$, the energy-momentum tensor is

$$\mathrm{T}_{\mu\nu} = \partial_\mu\varphi\partial_\nu\varphi - \frac{1}{2}g_{\mu\nu}(\partial_\alpha\varphi\partial^\alpha\varphi) - g_{\mu\nu}\,\mathrm{V}(\varphi)\,. \tag{4.10}$$

Inserting such matter into AdS_4, we have

$$\mathrm{R}_{\mu\nu} = -\frac{3}{\mathrm{L}^2}g_{\mu\nu} + \partial_\mu\varphi\partial_\nu\varphi + g_{\mu\nu}\mathrm{V}(\varphi)\,. \tag{4.11}$$

Now if t is the proper time of a lorentzian FRW spacetime obtained in this way, we find that the corresponding component of the Ricci tensor is

$$\mathrm{R}_{00} = \frac{3}{\mathrm{L}^2} + \dot{\varphi}^2 - \mathrm{V}(\varphi)\,. \tag{4.12}$$

We see that a *positive* $\mathrm{V}(\varphi)$ will *reduce* R_{00}; this *may or may not* lead — leaving aside the cosmological constant — to violations of the Strong Energy Condition. It will do so if the rate of evolution of the scalar field becomes sufficiently small; this often happens in a de Sitter-like spacetime, but *not necessarily* in an anti-de Sitter-like spacetime, where the Bang and the Crunch may keep the kinetic term sufficiently large so that the SEC is always satisfied. (Actually it does not necessarily occur even in the de Sitter-like case, as for example in some kinds of "eternal" quintessence.) In short, the status of the SEC in scalar field physics is ambiguous: it is violated in some circumstances but not in others.

Now let us consider the euclidean case. There is a very interesting ambiguity as to how a scalar field should be "euclideanized", particularly in the context of euclidean wormholes. It was argued by Giddings and Strominger [35] that an ordinary scalar field cannot generate topologically non-trivial euclidean configurations, such as wormholes, in the asymptotically flat context; instead they considered a *massless axion*, which, when expressed in terms of a locally equivalent single-component field, has a euclidean energy-momentum tensor of the opposite sign to that of the usual massless scalar euclidean energy-momentum tensor. As is emphasised in [35], this sign reversal is the key property needed to allow topologically non-trivial asymptotically flat configurations, for it allows the axion to make a negative contribution to the euclidean Ricci curvature. The theorem cited by Giddings and Strominger [36] does not apply here (among other things, the spaces we shall consider do not have any boundary of topology S^3, and, more importantly, they are not asymptotically flat), and in fact the situation considered by Witten and Yau is considerably more delicate than the one covered by that theorem. However, the fact that the Witten-Yau theorem attempts to rule out wormholes by assuming that matter sources make a positive contribution to the euclidean Ricci tensor, combined with the Giddings-Strominger observation that axions do *not* make such contributions in the case of the wormholes they considered, suggests strongly that some kind of axionic matter is relevant to the kind of cosmological wormholes proposed by Maldacena and Maoz.

We are thus led to consider quintessence-like cosmological matter fields, where the usual quintessence field is replaced by a "generalized axion". The requirements of cosmology will force us to consider unconventional potentials — which is what we shall mean by "generalized". lorentzian axions with conventional potentials have in fact been discussed as dark energy candidates (see for example [37, 38]), but they are not suitable for our purposes, since the effective axion potential is bounded, which does not seem natural when approaching a Bang or a Crunch.

The matter field we shall consider in the next section 5 shares with the Giddings-Strominger axion the ability (for a certain parameter range) to make a negative contribution to the euclidean Ricci curvature; it is in fact a generalized axion in the above sense. Let us therefore consider the euclidean geometry corresponding to a generalized axion; it is convenient to do this by taking equation (4.11), reversing the signature, and complexifying the scalar field. Then we have euclidean Ricci eigenvalues given by

$$\mathrm{Ric}_{(0)} = -\frac{3}{\mathrm{L}^2} - \dot{\varphi}^2 + \mathrm{V}_e(\varphi) \tag{4.13}$$

and

$$\mathrm{Ric}_{(1)} = \mathrm{Ric}_{(2)} = \mathrm{Ric}_{(3)} = -\frac{3}{\mathrm{L}^2} + \mathrm{V}_e(\varphi)\,, \tag{4.14}$$

where $\mathrm{V}_e(\varphi)$ is the euclidean potential.

Obviously the last three eigenvalues will always exceed $-3/\mathrm{L}^2$ if $\mathrm{V}_e(\varphi)$ is positive (which is certainly the case for the potential we shall consider below). Even $\mathrm{Ric}_{(0)}$ will do so if $\mathrm{V}_e(\varphi)$ is positive and sufficiently large; on the other hand, with this euclideanization, it can also fall below $-3/\mathrm{L}^2$ if the euclidean potential is too small. Thus, in sharp contrast to the Yang-Mills case, it is *possible* to satisfy the Witten-Yau inequalities with generalized axion matter — though it is also possible to violate them, in the Giddings-Strominger manner. However, the WY inequalities can only be satisfied with the aid of potentials of a kind which tend to *violate* the Strong Energy Condition in the lorentzian case, that is, with potentials which are positive.

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It is of course easy to arrange scalar potentials which violate the SEC: that is the point of quintessence. The euclidean version can then satisfy the Witten-Yau inequalities, and so — one would think — disconnected boundaries should be forbidden if we use such matter in a FRW cosmology with positively curved or flat spatial sections. (Recall that Cai and Galloway [21] extended the Witten-Yau theorem to the (scalar) flat case, and that in the cosmological application the spatial sections have the same geometry as the boundary components of the euclidean version.) If this were the case, then one could argue that disconnected boundaries are not a matter of concern, *since they apparently cannot arise in cosmological models with realistic (SEC-violating) matter content.*

Puzzling as it seems, this is wrong, however: evidently we have not yet fully understood the ability of the euclidean Maldacena-Maoz spaces to have disconnected boundaries. To see this, we shall explicitly construct a family of AdS-like quintessence cosmologies such that the euclidean boundary is disconnected even though the Witten-Yau inequalities are satisfied. These cosmologies have some independent interest.

5. Quintessence instead of Yang-Mills

As we saw, Maldacena and Maoz construct their spacetimes by introducing Yang-Mills matter into anti-de Sitter spacetime. Here we shall follow their example, but with a different choice of bulk matter. This bulk matter will be quintessence [17] with a particular choice of potential. Ultimately, as discussed in the previous section 4, we might wish to regard it as "generalized axion" matter, but for simplicity we shall at first present it as an ordinary quintessence field. The entire discussion will be lorentzian until further notice.

A popular choice of quintessence potential is obtained by combining exponentials of the scalar field φ, since these can be motivated both by fundamental physics and by astrophysical arguments. For example, such potentials arise naturally in supergravity — see the recent discussion [39] in connection with Cosmic Censorship — and also M-theory [18]–[20]. On the astrophysical side, in [40] and [41] potentials similar to the one below are used, and the observational consequences are explored; see [42] for an extensive list of references on the use of exponential potentials in cosmology.

Our objective here is not to find a model which is completely realistic; we merely seek an accelerating analogue of the spacetimes considered by Maldacena and Maoz. For our purposes, it is essential to obtain *exact* solutions for the metric, so that the geometric properties of the spacetime can be analysed precisely; thus we do not allow any other form of matter apart from the AdS_4 cosmological constant and the quintessence field. We choose a very simple quintessence potential which is approximately exponential for both large negative and large positive values of φ. (It is also well-behaved under complexification of φ, that is, it remains real and positive; this is important for the axionic interpretation when we eventually turn to the euclidean version.) As usual, the potential has two free parameters, one scaling φ and one scaling the overall potential. For later convenience we choose these two parameters, ϖ and ξ, so that the potential has the following form:

$$\mathrm{V}(\varphi) = \frac{3 - \varpi^{-1}}{\xi^2}\cosh^2\left(\sqrt{\frac{4\pi}{\varpi}}\varphi\right). \tag{5.1}$$

We shall always take ϖ to satisfy

$$\varpi > \frac{1}{3}\,, \tag{5.2}$$

so that $\mathrm{V}(\varphi)$ is *strictly positive*. The parametrization is chosen so that, in the appropriate limit ($\varpi \to \infty$), the potential tends to a constant; so this is the limit in which quintessence becomes a positive cosmological constant. Differentiating $\mathrm{V}(\varphi)$ we obtain after some elementary algebra the following identity between $\mathrm{V}(\varphi)$ and its first derivative:

$$\left(\frac{\mathrm{dV}}{\mathrm{d}\varphi}\right)^2 - \frac{16\pi}{\varpi}\,\mathrm{V}^2 + \frac{48\pi}{\varpi\xi^2}\left(1-\frac{1}{3\varpi}\right)\mathrm{V} = 0\,, \tag{5.3}$$

a relation which we shall use below.

We now propose to introduce this kind of matter into an AdS_4 background with cosmological constant $-3/\mathrm{L}^2$. We shall search for lorentzian FRW solutions of the Einstein equations with flat but compact spatial sections, so that the metric will have the general form

$$g^- = -dt^2 + A^2\,a(t)^2[d\theta_1^2 + d\theta_2^2 + d\theta_3^2], \tag{5.4}$$

as in equation (2.2) above, so that A measures the circumferences of the torus, and $a(t)$ is the scale factor. (As before, we want tori here because we wish ultimately to consider an AdS/CFT kind of scenario for cosmology, and it is preferable for that purpose that the boundary should be compact. Note however that toral (or, more generally, compact flat) spatial sections are natural in many cosmological models, as for example those considered in most brane gas models: see [43, 44] and their references.)

With the usual kinetic term, the density and pressure corresponding to φ are

$$\rho = \frac{1}{2}\,\dot{\varphi}^2 + \mathrm{V}(\varphi) \tag{5.5}$$

and

$$\mathrm{p} = \frac{1}{2}\,\dot{\varphi}^2 - \mathrm{V}(\varphi) \tag{5.6}$$

respectively, and so the Einstein equation for FRW spacetimes with flat spatial sections becomes

$$(\frac{\dot{a}}{a})^2 = \frac{8\pi}{3}\left[\frac{1}{2}\,\dot{\varphi}^2 + \mathrm{V}(\varphi)\ -\ \frac{3}{8\pi L^2}\right]. \tag{5.7}$$

The field equation for φ is

$$\ddot{\varphi} + 3\frac{\dot{a}}{a}\,\dot{\varphi} + \frac{\mathrm{dV}}{\mathrm{d}\varphi} = 0\,. \tag{5.8}$$

This may be usefully re-written as

$$\dot{\varphi}^2(\frac{\dot{a}}{a})^2 = \frac{1}{9}\left[\frac{\mathrm{d}}{\mathrm{d}\varphi}\left[\frac{1}{2}\dot{\varphi}^2 + \mathrm{V}(\varphi)\right]\right]^2\,. \tag{5.9}$$

Substituting this into the Einstein equation (5.7) we have

$$\left(\frac{\mathrm{d}}{\mathrm{d}\varphi}\left[\frac{1}{2}\dot{\varphi}^2 + \mathrm{V}(\varphi)\right]\right)^2 = 24\pi\dot{\varphi}^2\left[\frac{1}{2}\dot{\varphi}^2 + \mathrm{V}(\varphi) - \frac{3}{8\pi L^2}\right]. \tag{5.10}$$

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After eliminating the derivative of $\mathrm{V}(\varphi)$ using (5.3), we can regard this as a relation between φ and $\dot{\varphi}^2$ only, and it can be solved for the latter in terms of the former; inverting $\mathrm{V}(\varphi)$ we can in principle solve for $\dot{\varphi}^2$ in terms of $\mathrm{V}(\varphi)$. We can spare ourselves that onerous task by noticing the structural similarity of (5.10) with the identity (5.3), which suggests the simple ansatz

$$\dot{\varphi}^2 = \mathrm{KV}(\varphi)\,, \tag{5.11}$$

where K is a constant to be determined by comparing (5.10) with (5.3). We find that indeed this solves (5.10) *provided* that ξ is equated to $\sqrt{8\pi}\mathrm{L}$, that (5.2) holds, and that K is chosen to be $2/(3\varpi - 1)$. (We stress that this procedure only works if we strictly enforce the inequality (5.2); this has the unfortunate consequence that our subsequent formulae will not reflect the fact, visible in (5.1), that we recover AdS_4 when $\varpi = 1/3$, that is, when the scalar field is switched off.)

Thus by fixing the value of ξ (thereby reducing to a one-parameter subset of solutions), we can solve for $\dot{\varphi}^2$ in terms of $\mathrm{V}(\varphi)$, obtaining

$$\dot{\varphi}^2 = \frac{2}{3\varpi - 1}\mathrm{V}(\varphi)\,. \tag{5.12}$$

This no longer involves the scale factor and so it can be solved directly:

$$\varphi = \pm\sqrt{\frac{\varpi}{4\pi}}\cosh^{-1}(\sec(\frac{t}{\varpi \mathrm{L}}))\,, \tag{5.13}$$

where the sign agrees with that of t (so that $\varphi(t)$ is smooth — note that $\dot{\varphi}$ is never zero), and where $\cosh^{-1}$ is defined to be non-negative.

Equation (5.12) can also be used to find the scale factor. Substituting it into equations (5.5) and (5.6) we find expressions for the quintessence density and pressure in terms of the potential,

$$\rho = \frac{3\varpi}{3\varpi - 1}\mathrm{V}(\varphi) \tag{5.14}$$

and

$$\mathrm{p} = \frac{2 - 3\varpi}{3\varpi - 1}\mathrm{V}(\varphi)\,. \tag{5.15}$$

Now the fact that the overall energy-momentum tensor is divergenceless gives us the standard relation

$$\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + \mathrm{p}) = 0\,; \tag{5.16}$$

eliminating the time derivatives and using (5.14) and (5.15) we obtain from this

$$\frac{\mathrm{d}ln(\rho)}{\mathrm{d}ln(a)} = -\frac{2}{\varpi}\,. \tag{5.17}$$

This gives us

$$\rho = Ca^{(-2/\varpi)}\,, \tag{5.18}$$

where C is a constant which we can fix as follows. We are interested in obtaining Bang/ Crunch cosmologies. This means that there must be a time when the scale factor reaches a maximum and so has zero time derivative. From the Einstein equation (5.7) we see that, at this time, the total density (negative anti-de Sitter density plus positive quintessence density) must vanish. Because the spatial sections are flat, their intrinsic scale is not fixed by the other parameters so we are free to require that $a(t)$ should be equal to unity at this time. (This simply means that we are defining the parameter A in equation (5.4) to be such that the circumference of the *maximal* toral spatial section is 2πA.) Thus ρ must equal $3/8\pi \mathrm{L}^2$ when $a = 1$, and this fixes C at $3/8\pi \mathrm{L}^2$. Substituting (5.18) with this value of C back into the Einstein equation (5.7), we obtain a differential equation for $a(t)$:

$$(\frac{\dot{a}}{a})^2 = \frac{1}{L^2}[a^{(-2/\varpi)} - 1]\,. \tag{5.19}$$

This equation has an *exact* solution: defining t = 0 to be the time of maximum expansion, we have

$$a(t) = \cos^{\varpi}\left(\frac{t}{\varpi L}\right), \tag{5.20}$$

and so we arrive finally at a family of "quintessential Maldacena-Maoz" lorentzian metrics, parametrised by A and ϖ, given by the very simple metric

$$g^{-}(\varpi, A) = -dt^2 + A^2\cos^{2\varpi}\left(\frac{t}{\varpi \mathrm{L}}\right)[d\theta_1^2 + d\theta_2^2 + d\theta_3^2]\,. \tag{5.21}$$

Note that the metric $g^{-}(1, A)$, given by equation (2.2), which we obtained simply by replacing the negatively curved spatial sections of AdS_4 by flat tori, is indeed the special case $\varpi = 1$. The more general metric $g^{-}(\varpi, A)$ may be thought of in the same way: replace the negatively curved spatial sections of AdS_4 by flat spaces and replace $\cos^2(t/L)$ with $\cos^{2\varpi}(t/\varpi L)$.

The metrics $g^{-}(\varpi, A)$ apparently represent universes which have a Big Bang at $t = -\pi\varpi L/2$, expand to a maximum size at t = 0, and then collapse to a Big Crunch at $+\pi\varpi \mathrm{L}/2$. However, we know from the AdS_4 example that such appearances can be deceptive, so let us verify that these spacetimes are indeed singular.

The scalar curvature is given by

$$\mathrm{R}(g^{-}(\varpi, A)) = -\frac{12}{\mathrm{L}^2} + \frac{6}{\mathrm{L}^2}\left[2 - \frac{1}{\varpi}\right]\sec^2\left(\frac{t}{\varpi \mathrm{L}}\right), \tag{5.22}$$

which immediately shows that the metrics for different values of ϖ are distinct and that all of these metrics are singular at $\mathrm{t} = \pm\pi\varpi \mathrm{L}/2$, with the possible exception of $g^{-}(1/2, A)$, which can be shown to be singular in other ways. (Consider again equation (2.2): the metric $g^{-}(1, A)$ differs from that of AdS_4 only in that the spatial sections have been flattened. We now see that this flattening causes the spacetime to become singular, as claimed.)

We see that these are indeed Bang/Crunch cosmologies with a total lifetime given by $\pi\varpi$L. This number must of course be large, significantly larger than the current age of the Universe. As we shall soon see, observational evidence can in principle fix ϖ and L separately, and such data as we have suggest that ϖ must be large; for definiteness we shall assume that L is roughly equal to the age of the Universe, so that the factor $\pi\varpi$ is responsible for stretching the time scale. From the AdS/CFT point of view [45], the value of L in pure AdS_4 is related to the strength of the coupling of the boundary field theory. Here we do not have pure AdS_4, but we see from (5.22) that L continues to set the overall scale of the curvature; so we shall continue to assume that if these spacetimes have a holographic interpretation, then L is still a rough measure of the strength of the coupling in the dual field theory. By taking L to be of a typical cosmological size, we are implicitly assuming that the field theory is strongly coupled. We shall see that large values of ϖ lead to particularly interesting models.

The time-time component of the Ricci tensor is given by

$$\mathrm{R}_{00}(g^{-}(\varpi, A)) = \frac{3}{\mathrm{L}^2} - \frac{3}{\mathrm{L}^2}\left[1 - \frac{1}{\varpi}\right]\sec^2\left(\frac{t}{\varpi \mathrm{L}}\right). \tag{5.23}$$

Clearly quintessence always makes a non-negative contribution to this component of the Ricci tensor, that is, the Strong Energy Condition is satisfied at all times, if and only if $\varpi \leq 1$. For these values of ϖ, the cosmology is much like the traditional dust FRW model with density greater than the critical density: the Universe decelerates at all times from a Bang to a Crunch.

We shall be more interested in values of ϖ significantly greater than unity. For these, we see that the Strong Energy Condition is only satisfied for an interval of time around t = 0, namely the interval

$$|\mathrm{t}| \leq \varpi \mathrm{L} \cos^{-1}\sqrt{1 - \frac{1}{\varpi}}. \tag{5.24}$$

As a fraction of the total duration of the Universe, the length of this interval is

$$\Delta = \frac{2}{\pi}\cos^{-1}\sqrt{1 - \frac{1}{\varpi}}, \tag{5.25}$$

which is a decreasing function of ϖ; for very large ϖ, the era in which the SEC holds is a very small fraction of the total duration of the universe. Notice that this is independent of the value of L.

In order to show how the values of ϖ and L can in principle be fixed in terms of cosmological observations, we begin by noting that the current value of the Hubble constant in these cosmologies is given according to (5.19) and (5.20), by

$$\mathrm{H}_0 = \frac{1}{\mathrm{L}}\cot\left(\frac{\mathrm{T}}{\varpi \mathrm{L}}\right), \tag{5.26}$$

where T is the current age of the Universe, that is, measured from $\mathrm{t} = -\pi\varpi\mathrm{L}/2$; regarding T and H_0 as known from observations, we have one relation between ϖ and L.

To obtain another, observe that the equation-of-state parameter w for these cosmologies (that is, the ratio of the total pressure to the total density) is given, using equations (5.14) and (5.15), by

$$\mathrm{w} = \frac{\frac{3}{8\pi L^2} + \frac{2-3\varpi}{3\varpi}\rho}{-\frac{3}{8\pi L^2} + \rho}. \tag{5.27}$$

Using equation (5.18), with C fixed at $3/8\pi\mathrm{L}^2$, we can write this simply as

$$\mathrm{w} = -1 + \frac{2}{3\varpi}\mathrm{cosec}^2\left(\frac{t}{\varpi \mathrm{L}}\right); \tag{5.28}$$

as in pure quintessence scenarios, the theory cannot tolerate values of w below -1, and would be ruled out by firm evidence in favour of such values. Evaluating w at the present time, we have

$$\mathrm{w}_0 = -1 + \frac{2}{3\varpi}\sec^2\left(\frac{\mathrm{T}}{\varpi \mathrm{L}}\right), \tag{5.29}$$

where again T is the current age of the Universe. In principle, w_0 can be fixed by observation: in practice this can only be done rather roughly, but there is evidence [46] that it is close to -1, showing that ϖ is rather large, as we have been assuming. We now have two relations between ϖ and L, (5.26) and (5.29), so these parameters are fixed in terms of observed quantities. (Readers who prefer the more traditional deceleration parameter q_0 will find that it is given by

$$\mathrm{q}_0 = -1 + \frac{1}{\varpi}\sec^2\left(\frac{\mathrm{T}}{\varpi \mathrm{L}}\right), \tag{5.30}$$

so that $\mathrm{q}_0 = \frac{1}{2}(1 + 3\mathrm{w}_0)$, which is essentially just the Raychaudhuri equation for a FRW cosmology with flat spatial sections.)

The metric $g^-(\varpi, A)$ gives an accurate picture of spacetime during the period when the expansion has diluted ordinary matter and radiation to insignificance; that is, during the "quintessence-dominated era" and during the subsequent era when even quintessence is diluted and the geometry is dominated by the *negative* cosmological constant in the background. As we are ignoring ordinary matter and radiation here, $g^-(\varpi, A)$ cannot of course be expected to describe the present state of the Universe, and so at this stage it would be pointless to try to compute L and ϖ from current observational data. Purely for illustrative purposes we shall take it that $\mathrm{L} = \mathrm{T}$ and that $\pi\varpi = 30$; that is, the total lifetime of the Universe is assumed to be 30 times its current age. Then the graph of the scale function against t/T is shown in figure 2. With such values of ϖ, we see that the history of this universe is as follows.

Starting from the Big Bang at $\mathrm{t} = -\pi\varpi\mathrm{L}/2$, the SEC is immediately violated and the universe accelerates. (In reality of course there should be a period of deceleration — now observed [46] — but the absence of this early deceleration from this cosmology is simply due to our neglect of matter and radiation.) The expansion proceeds beyond the present time ($t/T = -14$ in figure 2) at a sedate pace, despite the fact that w is only slightly larger than -1 (equation (5.28)), until a large fraction of its total lifetime has passed. (This happens because, during this time, the scale factor is still very small, so the quintessence has not had the opportunity to cause a rapid expansion: in figure 2, the value of the scale function at the present time is roughly 4.31×10^{-10}.) This corresponds to the period in which, according to [46], we now find ourselves, with w close to -1 (it is equal to about -0.9294 at the present time under the conditions assumed in figure 2) and changing only very slowly. (See [5] for an alternative view of the observational data.) At some point, however, the rate of expansion increases dramatically; but w begins to rise until it reaches $-1/3$. (At this time, the scale factor is given, according to equations (5.20) and (5.28), by $(1 - 1/\varpi)^{\varpi/2}$, which is about $1/\sqrt{e}$ for large values of ϖ.) This is the point of transition from acceleration to deceleration. The equation-of-state parameter soon becomes positive

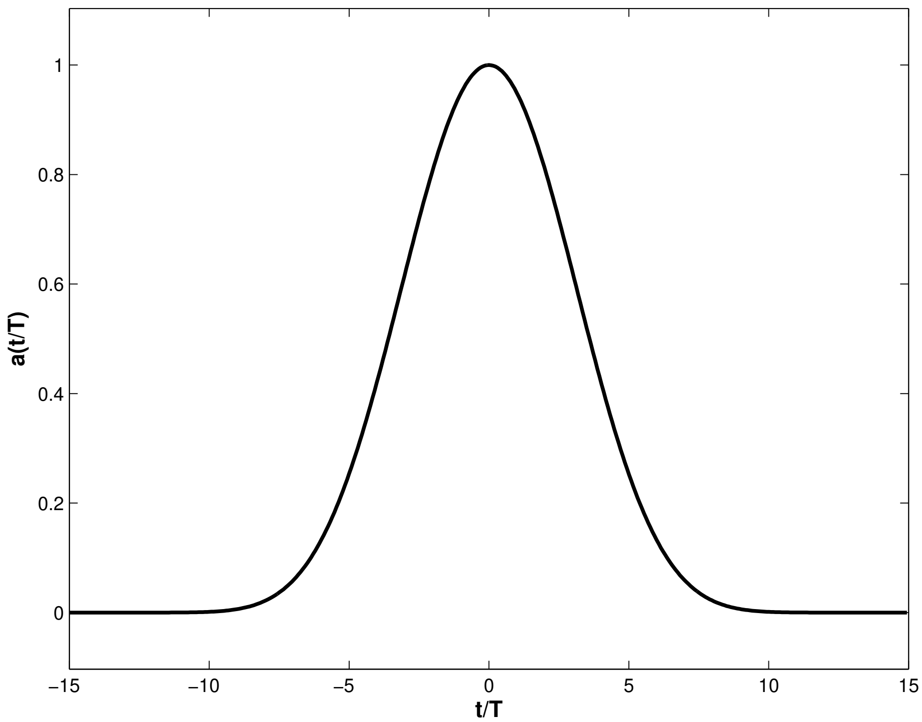

Figure 2: Quintessential MM scale function for L = T, $\pi\varpi$ = 30.

and in fact larger than +1; in this respect the situation is analogous to the one considered in [47], though of course that work is concerned with contraction to a minimum size instead of expansion to a maximum. The expansion halts at t = 0, and then a rapid contraction, still under the influence of the negative cosmological constant in the background, begins. There is then another transition back to an SEC-violating regime; although the period during which the universe is very large and decelerating is very short, by this time the universe is contracting so rapidly that a Big Crunch cannot be averted; it takes place at t $= \pi\varpi$L/2. It is interesting that the Universe ends its days in a futile effort, by accelerating again, to avert destruction. Observers at that time might be misled into believing that they live in a de Sitter-like world which might "bounce" and re-expand. The key point here is that only a brief period of deceleration is necessary to bring about a Crunch.

The causal structure of these spacetimes is rather interesting and relevant to our later discussion, so we consider it briefly. If we unwrap the spatial sections, so that they are copies of $\mathbb{R}^3$ instead of tori, then our spacetimes are conformal to Minkowski spacetime. Defining a parameter λ by $\cos(t/\varpi L) = sech(\lambda)$, we see that the full extent of conformal time is given by

$$\varpi \mathrm{L} \int_{-\infty}^{\infty} \cosh^{\varpi - 1}(\lambda) d\lambda \,, \tag{5.31}$$

which converges if $\varpi < 1$. The spatial sections being compact, the Penrose diagram in this relatively uninteresting case will be rather like the one given in figure 1 above. Since $\varpi < 1$ implies that the Strong Energy Condition is satisfied at all times, the similarity to that case is perhaps not very surprising. (The precise shape of the diagram in this case is, however, at our disposal, since the scale of the spatial tori can be fixed independently of all other parameters. Thus, the Penrose diagram can be either short or tall, just as we decide.)

In the much more interesting case $\varpi > 1$, in which the SEC does not hold at all times, the situation is quite different. The peculiar feature here is that both the Bang and the Crunch are infinitely remote in conformal time. This means that the Penrose diagram resembles that of Minkowski spacetime, but a singularity occupies all of the region $t = \pm\infty$ on the edges of the usual diamond, so that there are separate singularities for timelike and null geodesics. Thus the Bang is visible at all times, but the part of the Bang that one can see is not the part from which timelike geodesics emanate.

If we now compactify the spatial sections to tori, then there is no longer any spatial infinity or any null infinity. There is still an infinity for timelike geodesics, of course, but it is represented only by a pair of points in the diagram, one each for the Bang and the Crunch, shown as the heavy dots in figure 3. Essentially what happens is that null geodesics wind around the torus an infinite number of times as the Universe reaches the Crunch (or as they are traced back to the Bang); they cannot reach either singularity. In fact, the spacetime is null geodesically complete. Thus the Bang itself is dark.

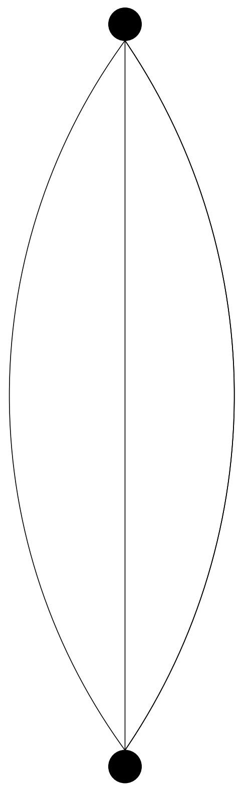

Figure 3: Penrose diagram of the Quintessential MM Cosmologies.

Since a torus is not globally isotropic, it is difficult to represent the situation in a fully adequate way on a Penrose diagram, but we can obtain a partly satisfactory picture by focusing on one of the coordinate directions θ_1,θ_2, or θ_3; let us pick θ_1. The vertical straight line in figure 3 represents $\theta_1 = 0$, and the two curved lines correspond to $\theta_1 = \pm\pi$. (That is, they indicate the topological identifications which define a torus. One can also, however, think of them as the timelike geodesic traced out by a point on a suppressed two-torus, which, like any other timelike geodesic, begins in the Bang and ends in the Crunch.)

One strange property of this cosmology is that null geodesics can circumnavigate the Universe, no matter how large the spatial tori may be. Indeed, at any point in this spacetime one can find a null geodesic which has performed an arbitrarily large number of circumnavigations. This is in sharp contrast to the case of figure 1; in that spacetime, circumnavigations are completely impossible (since the diagram is less than twice as high as it is wide). In principle, therefore, this cosmology predicts that the non-trivial topology of its spatial sections must be visible, and this appears to be in conflict with the observations [48]. However, the cosmic microwave radiation which we see today does not emanate from the Bang itself: one can think of the corresponding null geodesics as beginning on an initial spacelike hypersurface. If the present age of the Universe is sufficiently short relative to its total lifetime, and if the parameter A in (5.21) is chosen to be sufficiently large, then one can easily arrange for the past lightcone of the present moment to intersect this initial surface before the cone extends far enough to reveal the topology. Notice in this connection that there is no "isotropy problem" here, since the Bang is a single point in the Penrose diagram.

To summarize: we have a family of cosmological models which are obtained in much the same way as those considered by Maldacena and Maoz, but with flat spatial sections and with a transition from acceleration to deceleration, culminating in a Big Crunch. They seem to be physically acceptable, within the constraints imposed by the fact that we have not tried to set up a fully realistic matter model. Nevertheless the global structure of the lorentzian version differs very radically from that of the Maldacena-Maoz spacetime (figure 1). We shall return to this in the Conclusion.

However, the most puzzling feature of these spacetimes only appears when we turn to the euclidean version of (5.21), given by

$$g^{+}(\varpi, A) = dt^2 + A^2 \cosh^{2\varpi}\left(\frac{t}{\varpi \mathrm{L}}\right)\left[d\theta_1^2 + d\theta_2^2 + d\theta_3^2\right]. \qquad (5.32)$$

For large positive and negative t this is, for all ϖ, approximately

$$g^{+}(\varpi, A) \approx dt^2 + 4^{-\varpi} A^2 \exp\left(\frac{2|t|}{\mathrm{L}}\right)\left[d\theta_1^2 + d\theta_2^2 + d\theta_3^2\right], \qquad (5.33)$$

precisely as Maldacena and Maoz require (see equation (2.4)). Clearly $g^{+}(\varpi, A)$ is defined on a space with a conformal boundary which is compact but which consists, at least in the simplest interpretation, of two disconnected tori, just as the conformal boundary of the euclidean Maldacena-Maoz space consists of two disconnected spheres. The great difference between the two only becomes apparent when we compute the eigenvalues of the Ricci tensor of $g^{+}(\varpi, A)$. They are given by

$$\mathrm{Ric}_{(0)}(g^{+}(\varpi, A)) = -\frac{3}{\mathrm{L}^2} + \frac{3}{\mathrm{L}^2}\left[1 - \frac{1}{\varpi}\right] sech^2\left(\frac{t}{\varpi \mathrm{L}}\right) \qquad (5.34)$$

and

$$\mathrm{Ric}_{(1,2,3)}(g^{+}(\varpi, A)) = -\frac{3}{\mathrm{L}^2} + \frac{3}{\mathrm{L}^2}\left[1 - \frac{1}{3\varpi}\right] sech^2\left(\frac{t}{\varpi \mathrm{L}}\right). \qquad (5.35)$$

At once we see that our quintessence field (which always satisfies the inequality (5.2)) can never make a negative contribution to the last three eigenvalues. It *can* make a negative contribution to $\mathrm{Ric}_{(0)}(g^{+}(\varpi, A))$, *but only in the cosmologically least interesting case*, namely, when $\varpi < 1$. (The very fact that this is possible confirms that we are indeed dealing with a generalized axion here: see the discussion in the previous section 4.)

In the case of real interest, $\varpi > 1$, the Witten-Yau inequalities (3.2) are fully satisfied; *and yet the euclidean boundary is disconnected.* This seems to contradict the Giddings-Strominger requirement [35] that a wormhole should make a negative contribution to the euclidean Ricci curvature, but this is readily understood when we observe that these cosmological wormholes are not asymptotically flat. Less easy to understand is the apparent contradiction of the Witten-Yau theorem. We now explain how this is possible.

6. Escaping the menace of the WY theorem, part 2

In order to understand this puzzling situation, let us remind ourselves of the definition of a conformal boundary.

Let M^{n+1} be a non-compact $(n+1)$-dimensional manifold which can be regarded as the interior of a compact, connected manifold-with-boundary $\overline{\mathrm{M}}^{n+1}$, and let N^n be the boundary (which need not be connected). Let $g^+(\mathrm{M})$ be a smooth euclidean metric on M^{n+1} such that there exists a function G on $\overline{\mathrm{M}}^{n+1}$ with the following properties:

(*a*) $\mathrm{G(x)} = 0$ if and only if $\mathrm{x} \in \mathrm{N}^n$;

(*b*) $\mathrm{dG(x)} \neq 0$ for all $\mathrm{x} \in \mathrm{N}^n$;

(*c*) $\mathrm{G}^2 g^+(\mathrm{M})$ extends continuously to a metric on $\overline{\mathrm{M}}^{n+1}$;

(*d*) If $|\mathrm{dG}|$ is the norm of dG with respect to the extended metric, then $|\mathrm{dG}|$, evaluated on N^n, must not depend on position there.

Then we say that $\overline{\mathrm{M}}^{n+1}$ is a conformal compactification of M^{n+1}, that N is the conformal boundary, and that G is a defining function for N. (It is customary to impose various differentiability conditions on G, but we shall only require the existence of continuous first derivatives.)

The first three conditions ensure that the boundary is infinitely far from any point in the interior as measured by $g^+(\mathrm{M})$. The last point is not usually mentioned, because it is not necessary when the bulk is an Einstein space; however, we do need it here. It ensures that all sectional curvatures along geodesics "tending to infinity" approach a common negative constant, the asymptotic sectional curvature, which is equal to $-|\mathrm{dG}|^2$, evaluated on N^n. (Despite appearances, this is independent of the choice of G.) The metric is said to be *asymptotically hyperbolic* for this reason. It follows that the eigenvalues of the Ricci tensor must all approach $-3|\mathrm{dG}|^2$ in four dimensions.

For $g^+(\varpi, A)$, a natural choice for G can be constructed as follows. First define a constant c_ϖ by

$$\mathrm{c}_\varpi = \frac{\varpi}{\pi}\int_0^\infty sech^\varpi(\zeta)d\zeta\,; \tag{6.1}$$

the integral clearly converges, so c_ϖ is well defined; note that it depends only on ϖ. Now define a new coordinate θ by $\mathrm{c}_\varpi \mathrm{L} d\theta = \pm \mathrm{sech}^\varpi(\frac{t}{\varpi \mathrm{L}})\mathrm{dt}$, where the sign is chosen as + when t is positive, − when t is negative. An elementary calculation shows that the range of θ is just $-\pi$ to $+\pi$, corresponding to t ranging from $-\infty$ to $+\infty$; that is, the boundaries are at the finite θ values $\pm\pi$, and $\theta = 0$ corresponds to $\mathrm{t} = 0$. Now solve for t in terms of θ and use this to express $\mathrm{sech}^\varpi(\frac{t}{\varpi \mathrm{L}})$ in terms of θ. The resulting function, denoted $\mathrm{G}_\varpi(\theta)$, vanishes at $\pm\pi$, is at least once differentiable, and satisfies all of the conditions for a defining function for $g^+(\varpi, A)$. For example, the reader can verify that $\mathrm{G}_2(\theta)$ is given simply by $1-(\theta/\pi)^2$ and that it is a defining function for $g^+(2, A)$.

Using θ, we find that in the general case our metric is

$$g^+(\varpi, A) = \mathrm{G}_\varpi^{-2}(\theta)[c_\varpi^2 \mathrm{L}^2 d\theta^2 + A^2(d\theta_1^2 + d\theta_2^2 + d\theta_3^2)]\,. \tag{6.2}$$

Computing the norm of dG_ϖ with respect to $\mathrm{G}_\varpi^2(\theta)g^+(\varpi, A)$ (using of course the inverse metric to evaluate the norm of a one-form), we obtain

$$-\,|\mathrm{dG}_\varpi|^2 = -\frac{1}{\mathrm{L}^2}[1 - \mathrm{G}_\varpi^{2/\varpi}(\theta)]\,. \tag{6.3}$$

All of the above conditions are satisfied and the asymptotic sectional curvature (towards both connected components of the conformal boundary) is $-1/\mathrm{L}^2$; the Ricci eigenvalues must approach $-3/\mathrm{L}^2$, which indeed they do.

Now the Witten-Yau inequalities (3.2) have a straightforward meaning when the eigenvalues are *constants*, as of course they are when the bulk is an Einstein manifold. But when, as in the case considered by Maldacena and Maoz, the bulk is not an Einstein manifold, the Ricci eigenvalues are functions of position. It turns out that, in this situation, the Witten-Yau theorem needs not just (3.2) but also some restriction on the rate at which the Ricci eigenvalues approach their asymptotic value, $-3/\mathrm{L}^2$. This restriction was supplied by Cai and Galloway [21], and may be stated simply as follows in the general case.

Cai and Galloway express the metric of an asymptotically hyperbolic space, near to any connected component of the boundary, in the form

$$g^+(\mathrm{M}) = \frac{\mathrm{L}^2}{r^2}[dr^2 + g_r]\,. \tag{6.4}$$

The advantage of this way of writing the metric is that the coordinate r measures distance to the boundary according to the re-scaled metric (so that the boundary is at $r = 0$). Using r, we can measure the rate at which a given function $J(r)$ tends to zero as the boundary is approached — clearly $J(r)$ may be said to tend to zero very rapidly towards infinity if $J(r)/r^n$ tends to zero towards the boundary even for some large power n. In particular, suppose that the asymptotic sectional curvature is $-1/L^2$, so that the Ricci eigenvalues satisfy $\mathrm{Ric}_{(j)} + 3/\mathrm{L}^2 \rightarrow 0$ as conformal infinity is approached. The question now is *how quickly* these functions of position tend to zero.

We shall say that the *Cai-Galloway conditions* are satisfied for the conformal compactification of a four-manifold if

$$\mathrm{r}^{-2}\left[\mathrm{Ric}_{(j)} + \frac{3}{\mathrm{L}^2}\right] \rightarrow 0 \tag{6.5}$$

uniformly as infinity is approached, for all j. Cai and Galloway show, in a beautiful paper [21] using quite different techniques to those of [16], that if all of the other conditions of the Witten-Yau theorem are satisfied, including the inequalities (3.2), *and if the Cai-Galloway conditions are also valid*, then the conformal boundary must be connected.

Now in the case of the quintessential manifolds we have been considering, the metric $g^+(\varpi, A)$ can always be put in the form (6.4) near either component of the boundary by defining r by $t = \pm L\ln(r/L)$ (the choice of sign being determined by which component of the boundary we select); for then we can express the defining function $\mathrm{G}_\varpi(\theta)$ in terms of r as

$$\mathrm{G}_\varpi = 2^\varpi \frac{r}{\mathrm{L}}\left[1 + \left(\frac{r}{\mathrm{L}}\right)^{2/\varpi}\right]^{-\varpi}\,. \tag{6.6}$$

Now equations (5.34) and (5.35), expressed in terms of r, yield

$$r^{-2}\left[\mathrm{Ric}_{(0)}(g^{+}(\varpi,A))+\frac{3}{\mathrm{L}^{2}}\right] = \frac{12[\varpi-1](r/\mathrm{L})^{\frac{2[1-\varpi]}{\varpi}}}{\varpi \mathrm{L}^{4}[1+(r/\mathrm{L})^{\frac{2}{\varpi}}]^{2}}\,, \tag{6.7}$$

$$r^{-2}\left[\mathrm{Ric}_{(1,2,3)}(g^{+}(\varpi,A))+\frac{3}{\mathrm{L}^{2}}\right] = \frac{12\,[\varpi-1/3]\,(r/\mathrm{L})^{\frac{2[1-\varpi]}{\varpi}}}{\varpi \mathrm{L}^{4}[1+(r/\mathrm{L})^{\frac{2}{\varpi}}]^{2}}. \tag{6.8}$$

At once we see where the problem lies. Recalling that r tends to zero towards the boundary, we see that for values of ϖ strictly between 1/3 and 1, the Cai-Galloway conditions are satisfied (the right hand sides tend to zero) but the Witten-Yau inequalities are not. For values of ϖ greater than or equal to unity, the reverse is true. In short, there is no choice of ϖ which can satisfy all of the conditions required by Witten, Yau, Cai, and Galloway. *This is how the euclidean boundary can be disconnected for all of these cosmologies*: the contribution made by ("axionic") quintessence to the euclidean Ricci curvature is either of the wrong sign or it decays towards the boundary too slowly for the Witten-Yau-Cai-Galloway theorem to apply.

For the Maldacena-Maoz metric (3.4) we can again choose a new coordinate such that the boundary components are at finite values of the new coordinate, and then express $[(\alpha+\frac{1}{4})^{1/2}\cosh(2t/\mathrm{L})-\frac{1}{2}]^{-1/2}$ in terms of this new coordinate to obtain a defining function. Again g_{MM} can be written in the form (6.4) near either component of the boundary, and one can show, using equation (4.8) and the tracelessness of the energy-momentum tensor, that

$$r^{-2}\left[\mathrm{Ric}_{(0)}(g_{MM})+\frac{3}{\mathrm{L}^{2}}\right] = \frac{-3\alpha r^{2}}{\mathrm{L}^{6}[1/2(\alpha+1/4)^{1/2}\{1+(r/\mathrm{L})^{4}\}-r^{2}/2\mathrm{L}^{2}]^{2}} \tag{6.9}$$

$$r^{-2}\left[\mathrm{Ric}_{(1,2,3)}(g_{MM})+\frac{3}{\mathrm{L}^{2}}\right] = \frac{\alpha r^{2}}{\mathrm{L}^{6}[1/2(\alpha+1/4)^{1/2}\{1+(r/\mathrm{L})^{4}\}-r^{2}/2\mathrm{L}^{2}]^{2}}\,. \tag{6.10}$$

That is, the effect of the matter is of order r^4 towards infinity, a result which is not surprising in view of the fact that we are dealing with Yang-Mills matter. The conditions (6.5) are always satisfied in this case; in fact the rate of decay is more than sufficiently rapid to satisfy the Cai-Galloway conditions. Thus the Maldacena-Maoz space always (that is, for all values of α) violates the Witten-Yau inequalities and satisfies the Cai-Galloway conditions. (In fact, the behaviour of the Maldacena-Maoz space is very similar to that of the quintessential space with $\varpi = 1/2$, which also has a factor proportional to r^2 on the right hand sides of (6.7) and (6.8).)

To understand these results physically, notice first that the Maldacena-Maoz space and the quintessential space with $\varpi < 1$ violate the Witten-Yau inequalities precisely because they satisfy the SEC. Conversely, the quintessential spacetimes do temporarily violate the SEC (while still having a Bang and a Crunch) if the parameter ϖ exceeds unity, and this is exactly the condition which ensures that the Witten-Yau inequalities are satisfied in the euclidean case. In short, the observation of cosmic acceleration means that the Witten-Yau inequalities actually *are* a physically reasonable set of conditions in cosmology, not because they correspond to the SEC *but because they correspond to its violation.*

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On the other hand, the Cai-Galloway conditions are also physically well-justified *in some circumstances.* The theory of asymptotically anti-de Sitter spacetimes was developed in [33], in a way which generalises the usual theory of asymptotically flat spacetimes. The fall-off conditions for matter given there are that $\Omega^{-3}\mathrm{T}^i_j$, where Ω is a canonical conformal factor which tends to zero towards infinity and T^i_j is the (1,1) energy-momentum tensor, should have a smooth (not necessarily zero] limit at the boundary. This corresponds to requiring that the Ricci eigenvalues (in the euclidean version) should tend to zero at least as quickly as r^2 (since we are requiring the limit to be zero). Thus we see that the Cai-Galloway conditions are in fact a euclidean version of those normally employed in asymptotically anti-de Sitter physics. Thus the Cai-Galloway conditions are well-motivated in such situations.

However, [33] is explicitly concerned with generalizations of asymptotically flat spacetimes — that is, the applications intended are to black holes and similar localised phenomena. In cosmology we expect rates of decay towards boundaries which are slower, not faster, than the analogous rates for black holes. Indeed, quintessence is the canonical example of a form of matter which decays more slowly as the Universe expands than any ordinary form of matter. Recall that, in a FRW cosmology with a constant equation-of-state parameter w and a scale factor a, the density decays according to $\mathrm{a}^{-3(1+w)}$. Cosmic acceleration requires $\mathrm{w} < -1/3$, and so we see that, at least in the constant-w case, the condition that the Universe should accelerate is that the density should decay at a *maximum* rate of a^{-2}. Transferring this to the euclidean domain, and comparing the FRW metric with (6.4), we see that *we should not expect the Cai-Galloway conditions to be satisfied by the euclidean version of an accelerating cosmology*, though they should be perfectly reasonable for (say) euclidean versions of AdS black holes.

Thus we have finally uncovered the real underlying reason for the ability of our anti-de Sitter-like quintessence cosmologies to have euclidean versions with disconnected boundaries: it is that the case where these spacetimes accelerate is precisely the case where the euclidean versions violate the Cai-Galloway conditions. The quintessence simply decays towards the boundaries "too slowly".

We can summarize as follows. Cai and Galloway [21] showed that, in the non-Einstein case, connected euclidean boundaries can be ensured if the Witten-Yau inequalities are satisfied, *provided* that the functions $\mathrm{Ric}_{(j)}+3/\mathrm{L}^2$ decay towards infinity sufficiently rapidly. But in our anti-de Sitter quintessence cosmologies these functions do not decay so rapidly in the physically most interesting cases. Thus disconnected euclidean boundaries cannot be excluded, and in fact they do occur. More generally, we can expect that

(*a*) The Witten-Yau inequalities *will* be satisfied by the euclidean version of an anti-de Sitter-like cosmology which undergoes periods of acceleration, as our Universe does.

(*b*) Such cosmologies will generically have euclidean versions which violate the Cai-Galloway conditions; and this violation will permit (though not require) a disconnected conformal boundary.

In other words, it seems that *the obstruction to disconnected boundaries discovered by Witten and Yau is naturally evaded in the version of cosmological holography proposed by Maldacena and Maoz.*

7. Conclusion

Our objective in this work is to persuade the reader that situations like the one considered by Maldacena and Maoz [14], where the holographic picture (apparently) involves two boundaries but only one bulk, can in fact arise in a quasi-realistic cosmological setting obtained by introducing dark energy into AdS_4. We have tried to do this by identifying precisely how the quintessential spacetime (with metric given in equation (5.21)) is able to evade the Witten-Yau theorem and its extension due to Cai and Galloway [21]. The essential point turns out to be the fact that in cosmology one is not always entitled to prescribe very rapid asymptotic rates of decay of matter fields towards infinity. Dark energy, in particular, dilutes with the cosmic expansion very slowly, too slowly for the theorem to apply. In essence, it is the peculiar refusal of dark energy to dilute in the conventional way that permits the euclidean version of spacetime to have a disconnected boundary.

The present work is intended to be complementary to [14]. The latter was concerned with showing that it is possible to have a well-behaved field theory, on a disconnected boundary, which is dual to a known truncation of a compactified supergravity theory in the bulk. This is an important point, but it leads to cosmological models with no acceleration. Here we have a bulk, again with a disconnected euclidean boundary, which leads to a more realistic cosmology, but we lack as yet a dual description of the bulk matter. The fact that the euclidean version of our quintessence field reveals it to be a "generalized axion" is probably relevant here. Perhaps the ideas advanced in [18, 49, 20] may prove useful. It would also be valuable to have more examples of *exact* quintessence spacetime metrics (see the methods of [50], see also [51]) leading to euclidean spaces with multiple boundaries. It may be possible to learn something useful from manifolds with *more* than two boundary components.

Granting, as now seems likely, that there really are no physical or mathematical objections to Maldacena-Maoz cosmologies, one has to confront the original issue identified by Witten and Yau [16]: how can two apparently independent field theories (on the boundary) both be dual to the same bulk? It seems that either the two field theories, despite appearances, are not really independent (see for example [52]), or the bulk, despite appearances, is not really connected. The latter is hard to believe here, however, simply because the only natural place for the spacetime to split is at its point of maximum expansion, precisely when it is "most classical". Other, less obvious solutions of this puzzling problem are discussed in the conclusion of [14].

We close with some speculative remarks related to this issue.

First, the reader should be aware that the statement that equations (3.4) and (5.32) represent metrics defined on a space with *two* boundaries is not a mathematical fact; it is an interpretation of the structure of the metric. To understand this, recall that there is a natural choice of coordinate θ, running from $-\pi$ to $+\pi$, such that (5.32) can be re-expressed in the form (6.2). For simplicity let us choose the radii of the three-tori to be given by A $= \mathrm{c}_\varpi \mathrm{L}$. Then (6.2) becomes

$$g^+(\varpi, A) = c_\varpi^2 \mathrm{L}^2 \mathrm{G}_\varpi(\theta)^{-2} \left[d\theta^2 + d\theta_1^2 + d\theta_2^2 + d\theta_3^2\right] . \tag{7.1}$$

The "obvious" interpretation of *this* is that the euclidean version of the quintessential metric is defined on an open subset, corresponding to the open interval $(-\pi, +\pi)$ for θ, in the four-dimensional torus $\mathrm{S}^1 \times \mathrm{T}^3 = \mathrm{T}^4$. A conformal transformation which strips away the factor $c_\varpi^2 \mathrm{L}^2 \mathrm{G}_\varpi(\theta)^{-2}$ allows us to extend the metric to the usual one on the cubic four-torus with all four circumferences equal to 2π. (Other choices of A simply yield non-cubic tori.) The conformal factor tends to infinity as we approach either $\theta = -\pi$ or $\theta = +\pi$, *but this is one three-torus, not two*; it is the same three-torus being approached from opposite sides. Similarly, the euclidean Maldacena-Maoz metric can be regarded as being defined on an open submanifold of $\mathrm{S}^1 \times \mathrm{S}^3$ or $\mathrm{S}^1 \times \mathbb{R}\mathrm{P}^3$: we have from (3.4)

$$g_{MM}^+ = \mathrm{G}_{MM}(\psi)^{-2} \left[\mathrm{b}_\alpha^2 \mathrm{L}^2 d\psi^2 + \mathrm{L}^2 (d\chi^2 + \sin^2(\chi)\{d\theta^2 + \sin^2(\theta) d\phi^2\})\right] , \tag{7.2}$$

where b_α is a constant, depending only on α, defined as

$$\mathrm{b}_\alpha = \frac{1}{2\pi} \int_0^\infty \frac{d\zeta}{\sqrt{(\alpha + 1/4)^{1/2} \cosh(\zeta) - 1/2}} , \tag{7.3}$$

where ψ is an angular coordinate on the circle S^1, ranging between $\pm\pi$, defined by $\mathrm{b}_\alpha \mathrm{L} d\psi = \pm[(\alpha + 1/4)^{1/2} \cosh(2t/\mathrm{L}) - 1/2]^{-1/2} \mathrm{d}t$, and where $\mathrm{G}_{MM}(\psi)$ is the defining function. ((Notice that b_α and (therefore) ψ are well-defined *unless* α is exactly zero.) Again the conformally deformed version extends to *all* of $\mathrm{S}^1 \times \mathrm{S}^3$ or $\mathrm{S}^1 \times \mathbb{R}\mathrm{P}^3$, with the circle being of radius $\mathrm{b}_\alpha \mathrm{L}$.

Of course this means that we are thinking of the conformal compactification space as a compact *manifold* instead of a compact manifold-with-boundary; here, infinity corresponds to a special submanifold instead of a pair of boundary components. (It is a simple exercise to adapt the usual definition of a conformal compactification to a definition in terms of infinitely distant *submanifolds* instead of *boundaries*.)

The point to be stressed is that one cannot *prove* mathematically that either of these interpretations — "two boundaries" or "one submanifold" — is the "correct" one. Equation (6.2) is after all obtained from (5.32) simply by changing a coordinate, and (7.1) is just a special case of (6.2). The question as to whether there really are two field theories here is a physical, not a mathematical one: it cannot be decided simply by inspecting the metric. If (6.2) had been found first, then we might have argued that the "toral" interpretation which it suggests is more natural than the "cylindrical" one suggested by (5.32), and then the "double-boundary" conundrum *would never have arisen.* Note that the boundary of *every* compact manifold-with-boundary can be interpreted as a submanifold of a compact *manifold*: simply take two copies of the manifold-with-boundary and identify them along the boundary. Thus the "submanifold" interpretation of conformal infinity is quite as general as the more familiar "boundary" interpretation. Indeed, in the case of the euclidean Maldacena-Maoz space with S^3 sections, we can see how this works explicitly: writing the

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metric as in equation (7.2), let α tend to zero. Then from equation (7.3), this causes b_α to diverge: the circle "snaps" as the wormhole pinches off and we are left with two copies of euclidean AdS_4 identified along a common boundary. (However, this would lead to a new problem: two bulks for one field theory.)

These observations allow us to formulate the "double-boundary problem" in a more physical way: the problem is connected with the question as to whether we can find a physical reason for preferring manifolds-with-boundary to manifolds. The situation here is very much analogous to the interpretational ambiguities of the Randall-Sundrum models [53, 54]: does the bulk extend away from the brane-world on both sides, or only on one? In the first case, the brane is a submanifold and *not* a boundary, as it is in the second case. If further investigations of brane-world models suggest that branes at "the end of the world" are physically unacceptable, then this might well extend to a physical argument showing that the "circular" interpretations of $g^+(\varpi, A)$ and g^+_{MM} suggested by (6.2) and (7.2) are the correct ones. This prompts a difficult question: what would this conclusion imply for the lorentzian versions of these spaces?

As is well known, the conformal compactification of lorentzian de Sitter spacetime also has an apparently double boundary, and this is the case also in asymptotically de Sitter "bouncing" cosmological models. The four-dimensional de Sitter metric can be written in globally valid coordinates as

$$g(dS_4) = \frac{L^2}{\cos^2(\psi/2)}\left[-\frac{1}{4}d\psi^2 + d\chi^2 + \sin^2(\chi)\{d\theta^2 + \sin^2(\theta)d\phi^2\}\right], \tag{7.4}$$

where ψ is again an angular coordinate running from $-\pi$ to $+\pi$. The fact that the conformal factor is a periodic function makes a topological identification of the boundary components, again converting a manifold-with-boundary to a compact manifold, particularly natural from a mathematical point of view. Physically, too, the identification seems natural, because the rapid contraction/expansion of the universe in such models at very early/late times does indeed tend to render physical conditions identical towards both components of infinity. (Such an identification of the conformally related spacetime does not, of course, violate causality in the original spacetime, since the closed timelike worldlines are infinitely long there.)

In the cases discussed here, however, it is hard to use such arguments because the boundaries we are discussing are boundaries of the *euclidean* versions of our spacetimes. Particularly in the case of $g^\pm(\varpi, A)$ with $\varpi > 1$, it is far from clear that it is correct to relate physical conditions near the singularities in figure 3 to conditions near the boundaries of the euclidean version. If it were correct to do so, then identifying the connected components of the euclidean boundary would (presumably) entail identifying the singularities at the top and bottom of figure 3. While this does not violate causality, it might do so if the singularities were somehow resolved. Perhaps Nature relies on cosmological singularities to preserve causality.

This discussion leads to our second observation. One striking feature of all of the spacetimes discussed here is that they seem to be holographic *only* in their euclidean versions. This is in contrast to AdS_5, and to some extent also to de Sitter spacetime dS_4.

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Let us explain. The conformal boundary of the (simply connected) lorentzian version of AdS_5 has the structure $\mathbb{R}\times \mathrm{S}^3$, and this is a suitable background for a lorentzian field theory. The euclidean version has S^4 as its boundary, and this again is a suitable domain for a euclidean field theory. For dS_4, the situation is less satisfactory: in the lorentzian case, the boundary consists of two copies of *euclidean* S^3, and this of course is part of the reason for the fact that it is difficult to make the dS/CFT correspondence [55] work as effectively as AdS/CFT. The euclidean version of dS_4 is normally taken to be the four-sphere S^4 (but see [56]), which has no holographic dual whatever since it has no boundary. When we consider the spacetimes discussed here, we find that the Maldacena-Maoz spacetime and our quintessential spacetime for $\varpi < 1$ both have Penrose diagrams like the one in figure 1. These do have extended boundaries, though not of the same kind as those of the lorentzian versions of AdS_5 and dS_4. However, they only have such Penrose diagrams because they never violate the SEC. In the more realistic case where there are periods of acceleration, that is, in the quintessential case with $\varpi > 1$, the Penrose diagram is as in figure 3. In this case there is no hope of establishing a holographic duality with a boundary consisting of the two singular points in that diagram. It would be very interesting to know whether the situation depicted in figure 3 is in some sense generic for anti-de Sitter-like cosmologies which, on the one hand, have well-behaved euclidean versions and which, on the other, can accommodate periods of acceleration together with a relatively short period of rapid deceleration, as in figure 2. If this is so, it suggests that holography in cosmology may be a strictly euclidean phenomenon.

Acknowledgments

The author is grateful for the kind hospitality of the High Energy Section of the Abdus Salam International Centre for Theoretical Physics, where most of this work was done. He also wishes to thank Greg Galloway for very helpful correspondence, Ishwaree Neupane for useful discussions, and the referee for the observation that our quintessence field may be some kind of axion. He is also grateful to Soon Wanmei for the diagrams and indeed for everything.

References

[1] S.M. Carroll, *Why is the universe accelerating?*, *ECONF* **C0307282** (2003) TTH09 [astro-ph/0310342].

[2] E.V. Linder, *Probing gravitation, dark energy and acceleration*, astro-ph/0402503.

[3] F. Larsen and R. McNees, *Holography, diffeomorphisms and scaling violations in the CMB*, hep-th/0402050.

[4] R. Bousso, *Adventures in de Sitter space*, hep-th/0205177.

[5] U. Alam, V. Sahni, T.D. Saini and A.A. Starobinsky, *Is there supernova evidence for dark energy metamorphosis?*, astro-ph/0311364.

[6] S. Nesseris and L. Perivolaropoulos, *A comparison of cosmological models using recent supernova data*, astro-ph/0401556.

[7] M. Dine and N. Seiberg, *Is the superstring weakly coupled?*, *Phys. Lett.* **B 162** (1985) 299.

[8] S. Kachru, R. Kallosh, A. Linde and S.P. Trivedi, *de Sitter vacua in string theory*, *Phys. Rev.* **D 68** (2003) 046005 [hep-th/0301240].

[9] M. Becker, G. Curio and A. Krause, *de Sitter vacua from heterotic M-theory*, hep-th/0403027.

[10] S.B. Giddings, *The fate of four dimensions*, *Phys. Rev.* **D 68** (2003) 026006 [hep-th/0303031].

[11] R.R. Caldwell, *A phantom menace?*, *Phys. Lett.* **B 545** (2002) 23 [astro-ph/9908168].

[12] T.R. Choudhury and T. Padmanabhan, *A theoretician's analysis of the supernova data and the limitations in determining the nature of dark energy, II. Results for latest data*, astro-ph/0311622.

[13] R. Kallosh, J. Kratochvil, A. Linde, E.V. Linder and M. Shmakova, *Observational bounds on cosmic doomsday*, *JCAP* **10** (2003) 015 [astro-ph/0307185].

[14] J. Maldacena and L. Maoz, *Wormholes in AdS*, *J. High Energy Phys.* **02** (2004) 053 [hep-th/0401024].

[15] B. McInnes, *The dS/CFT correspondence and the big smash*, *J. High Energy Phys.* **08** (2002) 029 [hep-th/0112066].

[16] E. Witten and S.T. Yau, *Connectedness of the boundary in the AdS/CFT correspondence*, *Adv. Theor. Math. Phys.* **3** (1999) 1635 [hep-th/9910245].

[17] P.J.E. Peebles and B. Ratra, *The cosmological constant and dark energy*, *Rev. Mod. Phys.* **75** (2003) 559 [astro-ph/0207347].

[18] P.K. Townsend, *Quintessence from M-theory*, *J. High Energy Phys.* **11** (2001) 042 [hep-th/0110072].

[19] I.P. Neupane, *Accelerating cosmologies from exponential potentials*, hep-th/0311071.

[20] L. Jarv, T. Mohaupt and F. Saueressig, *Quintessence cosmologies with a double exponential potential*, hep-th/0403063.

[21] M.L. Cai and G.J. Galloway, *Boundaries of zero scalar curvature in the AdS/CFT correspondence*, *Adv. Theor. Math. Phys.* **3** (1999) 1769 [hep-th/0003046].

[22] M. Visser, *Lorentzian wormholes from Einstein to Hawking*, AIP Press, 1995.

[23] B.T. McInnes, *A positive cosmological constant in string theory through AdS/CFT wormholes*, *Nucl. Phys.* **B 609** (2001) 325 [hep-th/0105151].

[24] G.W. Gibbons, *Anti-de-Sitter spacetime and its uses*, *Lecture Notes in Phys.* **537** (2000) 102.

[25] S.W. Hawking, G.F.R. Ellis, *The large scale structure of space-time*, Cambridge University Press, 1973.

[26] M. Cvetič, S. Nojiri and S.D. Odintsov, *Cosmological anti-de Sitter space-times and time-dependent AdS/CFT correspondence*, *Phys. Rev.* **D 69** (2004) 023513 [hep-th/0306031].

[27] E. Witten, *Anti-de Sitter space and holography*, *Adv. Theor. Math. Phys.* **2** (1998) 253 [hep-th/9802150].

[28] Michael T. Anderson, *Boundary regularity, uniqueness and non-uniqueness for AH Einstein metrics on 4-manifolds*, *Adv. Math.* **179** (2003) 205, math.DG/0104171.

[29] M.T. Anderson, *Geometric aspects of the AdS/CFT correspondence*, hep-th/0403087.

[30] Xiaodong Wang, *On conformally compact Einstein manifolds*, *Math. Res. Lett.* **8** (2001) 671.

[31] N. Seiberg and E. Witten, *The D1/D5 system and singular CFT*, *J. High Energy Phys.* **04** (1999) 017 [hep-th/9903224].

[32] A. Buchel, *Gauge theories on hyperbolic spaces and dual wormhole instabilities*, hep-th/0402174.

[33] A. Ashtekar, A. Magnon, *Asymptotically anti-deSitter space-times*, *Class. and Quant. Grav.* **1** (1984) L39.

[34] B. McInnes, *de Sitter and Schwarzschild-de Sitter according to Schwarzschild and de Sitter*, *J. High Energy Phys.* **09** (2003) 009 [hep-th/0308022].

[35] S.B. Giddings and A. Strominger, *Axion induced topology change in quantum gravity and string theory*, *Nucl. Phys.* **B 306** (1988) 890.

[36] J. Cheeger, D. Grommoll, Ann. Math. (2) 96 (1972) 413.

[37] K. Choi, *String or M-theory axion as a quintessence*, *Phys. Rev.* **D 62** (2000) 043509 [hep-ph/9902292].

[38] J.E. Kim and H.P. Nilles, *A quintessential axion*, *Phys. Lett.* **B 553** (2003) 1 [hep-ph/0210402].

[39] M. Gutperle and P. Kraus, *Numerical study of cosmic censorship in string theory*, *J. High Energy Phys.* **04** (2004) 024 [hep-th/0402109].

[40] R. Cardenas, T. Gonzalez, O. Martin and I. Quiros, *A model of the universe including dark energy accounted for by both a quintessence field and a (negative) cosmological constant*, *Phys. Rev.* **D 67** (2003) 083501 [astro-ph/0206315].

[41] C. Rubano, P. Scudellaro, E. Piedipalumbo and S. Capozziello, *Oscillating dark energy: a possible solution to the problem of eternal acceleration*, *Phys. Rev.* **D 68** (2003) 123501 [astro-ph/0311535].

[42] A. Kehagias and G. Kofinas, *Cosmology with exponential potentials*, gr-qc/0402059.

[43] S. Watson and R. Brandenberger, *Linear perturbations in brane gas cosmology*, *J. High Energy Phys.* **03** (2004) 045 [hep-th/0312097].

[44] T. Battefeld and S. Watson, *Effective field theory approach to string gas cosmology*, hep-th/0403075.

[45] O. Aharony, S.S. Gubser, J.M. Maldacena, H. Ooguri and Y. Oz, *Large-N field theories, string theory and gravity*, *Phys. Rept.* **323** (2000) 183 [hep-th/9905111].

[46] A.G. Riess et al., *Type ia supernova discoveries at $z > 1$ from the hubble space telescope: evidence for past deceleration and constraints on dark energy evolution*, astro-ph/0402512.

[47] J.K. Erickson, D.H. Wesley, P.J. Steinhardt and N. Turok, *Kasner and mixmaster behavior in universes with equation of state $w \geq 1$*, *Phys. Rev.* **D 69** (2004) 063514 [hep-th/0312009].

[48] N.J. Cornish, D.N. Spergel, G.D. Starkman and E. Komatsu, *Constraining the topology of the universe*, astro-ph/0310233.

[49] D. Bak, M. Gutperle and S. Hirano, *A dilatonic deformation of AdS_5 and its field theory dual*, *J. High Energy Phys.* **05** (2003) 072 [hep-th/0304129].

[50] J.G. Russo, *Exact solution of scalar-tensor cosmology with exponential potentials and transient acceleration*, hep-th/0403010.

[51] C.M. Chen, P.M. Ho, I.P. Neupane and J.E. Wang, *A note on acceleration from product space compactification*, *J. High Energy Phys.* **07** (2003) 017 [hep-th/0304177].

[52] V. Balasubramanian, J. de Boer and D. Minic, *Exploring de Sitter space and holography*, *Class. and Quant. Grav.* **19** (2002) 5655 [hep-th/0207245].

[53] L. Randall and R. Sundrum, *A large mass hierarchy from a small extra dimension*, *Phys. Rev. Lett.* **83** (1999) 3370 [hep-ph/9905221].

[54] L. Randall and R. Sundrum, *An alternative to compactification*, *Phys. Rev. Lett.* **83** (1999) 4690 [hep-th/9906064].

[55] A. Strominger, *The dS/CFT correspondence*, *J. High Energy Phys.* **11** (2001) 034, [hep-th/0106099].

[56] B. McInnes, *Exploring the similarities of the dS/CFT and AdS/CFT correspondences*, *Nucl. Phys.* **B 627** (2002) 311 [hep-th/0110062].

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